10. DNS, LES, and RANS

~ attempts to predict turbulence

$$\nabla \cdot \vec{u} = \frac{\partial u_j}{\partial x_j} = 0$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial u_i}{\partial x_j \partial x_j} + f_i$$

~ four equations for four unknowns u_j and p

DNS (Direct Numerical Simulation):

~ solve Navier-Stokes equations directly without modeling

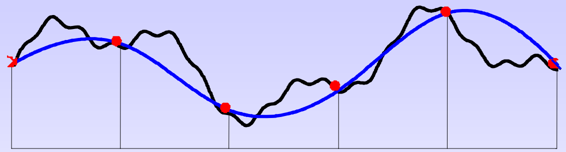
~ finest grid Δx ~ Kolmogorov's dissipation length scale

~ time increment Δt ~ Kolmogorov's dissipation time scale

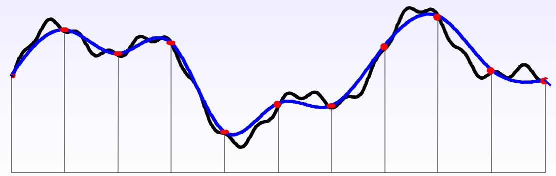
~ simulation time period = several L/q's

10. DNS, LES, and RANS

Resolution 1:

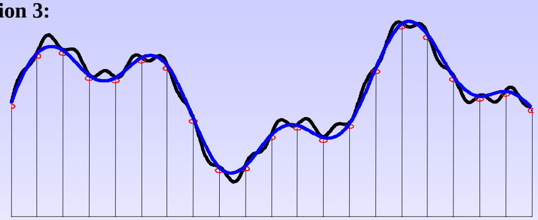


Resolution 2:

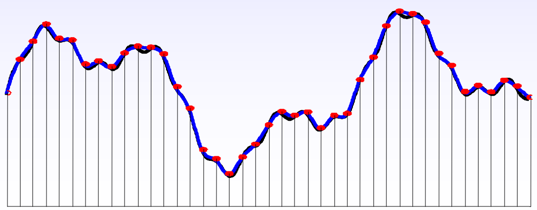


10. DNS, LES, and RANS

Resolution 3:



Resolution 4:



10. DNS, LES, and RANS

Estimation of the number of grids required for a DNS with $Re_L = 10^4$:

Kolmogorov's dissipation length scale ~ $\eta = L\, Re_L^{-3/4}$

$$\# \sim (L/\eta)^3 \sim Re_L^{9/4} \sim 10^9$$

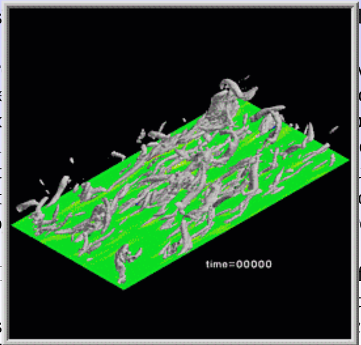
Estimation of the number of time steps required for a DNS with $Re_L = 10^4$:

Kolmogorov's dissipation time scale ~ $\tau_\eta = (\nu/\varepsilon)^{1/2} = \frac{L}{q} Re_L^{-1/2}$

$$\# \sim \frac{L/q}{\tau_\eta} \sim Re_L^{1/2} \sim 100$$

10. DNS

This animation demonstrates the evolution of **near-wall coherent structures** as ... leveloped turbulent channel flow. ... fluid is "spinning" fastest: such r... vn as "isosurfaces") which enclose ... discriminant of the **velocity grad... positive threshold**, indicating flui... egions also roughly (but not exact... uid is lowest, as do the visible co... ons above roughly correspond to ... **es** in the turbulent flow.

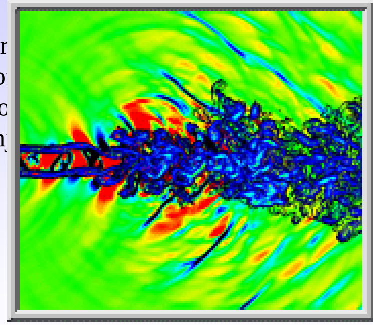


Only one eig... for this simulation, which used ... **ts**, is shown. The animation is s... ts at approximately 2/3 of the bulk velocity to slow the apparent movement of these structures so that their evolution may be more easily observed.

Simulation by Thomas Bewley (UCSD). Visualization by Ned Hammond (Stanford).

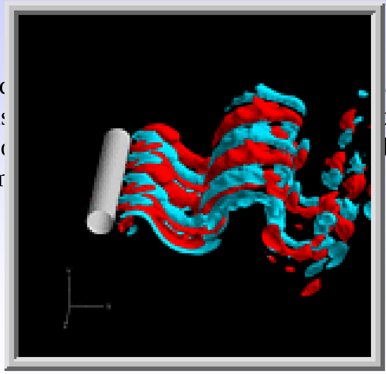
10. DNS

Color contour... rect Numerical Simulations o... acoustic field. The simulatio... Freund, Parviz Moin and San...



10. DNS

Color surfac... ative) of the instantaneous... ct Numerical Simulations ... he simulations were perform...

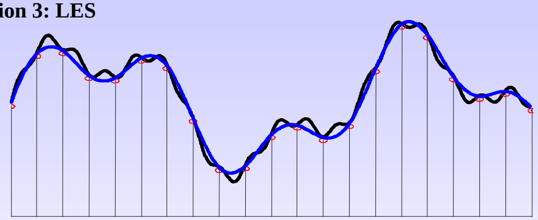


10. DNS

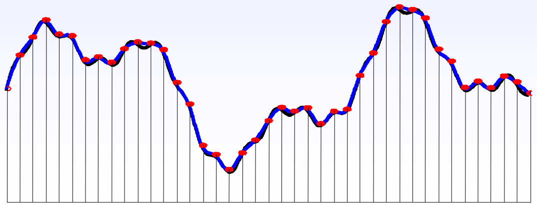
a spatially evolving free shear layer
Bert Debusschere

10. DNS, LES, and RANS

Resolution 3: LES



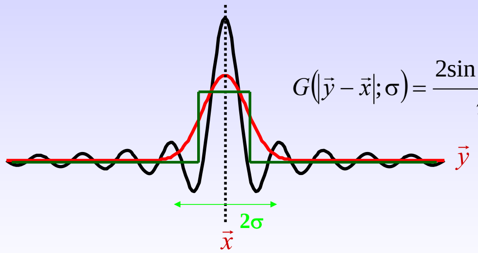
Resolution 4: DNS



10. LES

LES (Large Eddy Simulation) ~ spatial average

1. select a space filter $G(|\vec{y}-\vec{x}|;\sigma)$



$$G(|\vec{y}-\vec{x}|;\sigma)=\frac{2\sin(\pi|\vec{y}-\vec{x}|/\sigma)}{\pi|\vec{y}-\vec{x}|}$$

$$G(|\vec{y}-\vec{x}|;\sigma)=\frac{1}{\sqrt{2\pi}\sigma}\exp(-|\vec{y}-\vec{x}|^2/2\sigma^2)$$

$$G(|\vec{y}-\vec{x}|;\sigma)=\begin{cases}1/\sigma & \text{if } |\vec{y}-\vec{x}|\le\sigma/2;\\ 0 & \text{otherwise.}\end{cases}$$

10. LES

2. define large-eddy quantity:

$$\bar{u}_i(\vec{x},t)\equiv\int u_i(\vec{y},t)G(|\vec{y}-\vec{x}|;\sigma)d\vec{y}$$

$$u_i(\vec{x},t)=\bar{u}_i(\vec{x},t)+u_i'(\vec{x},t)$$

$\overline{\bar{u}_i}$	$\neq$	$\bar{u}_i$
$\overline{u_i'}$	$\neq$	0

$$\overline{\frac{\partial u(\vec{x},t)}{\partial x_j}}=\int\frac{\partial u(\vec{y},t)}{\partial y_j}G(|\vec{y}-\vec{x}|;\sigma)d\vec{y}=\frac{\partial}{\partial x_j}\int u(\vec{y},t)G(|\vec{y}-\vec{x}|;\sigma)d\vec{y}=\frac{\partial\bar{u}(\vec{x},t)}{\partial x_j}$$

$$\overline{\frac{\partial u(\vec{x},t)}{\partial t}}=\frac{\partial\bar{u}(\vec{x},t)}{\partial t}$$

Leonard stresses, L_{ij}

$$\overline{u_iu_j}=\bar{u}_i\bar{u}_j+\overline{\bar{u}_iu_j'}+\overline{\bar{u}_ju_i'}+\overline{u_i'u_j'}+\overline{\bar{u}_i\bar{u}_j}-\bar{u}_i\bar{u}_j$$

subgrid stresses, η_{ij}

10. LES

3. solve large-eddy motion $\bar{u}_i$:

$$\nabla\cdot\vec{u}=\frac{\partial u_j}{\partial x_j}=0$$

$$\frac{\partial u_i}{\partial t}+\frac{\partial(u_iu_j)}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2u_i}{\partial x_j\partial x_j}+f_i$$

~ four equations for four unknowns u_j and p

$$\frac{\partial\bar{u}_j}{\partial x_j}=0$$

$$\frac{\partial\bar{u}_i}{\partial t}+\frac{\partial(\bar{u}_i\bar{u}_j)}{\partial x_j}=-\frac{\partial}{\partial x_i}\left(\frac{\bar{p}}{\rho}+\tfrac{1}{3}\eta_{kk}\right)+\nu\frac{\partial^2\bar{u}_i}{\partial x_j\partial x_j}+\bar{f}_i+\frac{\partial}{\partial x_j}\left(\tau_{ij}^{sub}-L_{ij}\right)$$

$$\tau_{ij}^{sub}\equiv-\left(\eta_{ij}-\tfrac{1}{3}\eta_{kk}\delta_{ij}\right)$$ **traceless**

10. LES

4. Subgrid-stress models ~ modeling small-scale turbulence

$$\eta_{ij} \equiv \overline{\bar{u}_i u'_j} + \overline{\bar{u}_j u'_i} + \overline{+u'_i u'_j}$$

$$\tau_{ij}^{sub} \equiv -\left(\eta_{ij} - \tfrac{1}{3}\eta_{kk}\delta_{ij}\right)$$

~ effects of small eddies on large eddies through interactions in between

$$L_{ij} \equiv \overline{\bar{u}_i \bar{u}_j} - \bar{u}_i \bar{u}_j$$

~ effects of small eddies generated by large eddies on large eddies

~ theoretically computable (no need in modeling)

~ usually modeled together with subgrid stresses because its magnitude is usually on the order of truncation errors. In that case,

$$\tau_{ij}^{sub} \equiv -\left(\eta_{ij} - \tfrac{1}{3}\eta_{kk}\delta_{ij}\right) - \left(L_{ij} - \tfrac{1}{3}L_{kk}\delta_{ij}\right)$$

10. LES

$$\frac{\partial \bar{u}_j}{\partial x_j} = 0$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial(\bar{u}_i \bar{u}_j)}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(\frac{\bar{p}_{\text{mod}}}{\rho}\right) + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} + \bar{f}_i + \frac{\partial \tau_{ij}^{sub}}{\partial x_j}$$

$$\tau_{ij}^{sub} \equiv -\left(\eta_{ij} + L_{ij}\right) - \tfrac{1}{3}\left(\eta_{kk} + L_{kk}\right)\delta_{ij}$$

$$\frac{\bar{p}_{\text{mod}}}{\rho} \equiv \frac{\bar{p}}{\rho} + \tfrac{1}{3}\left(\eta_{kk} + L_{kk}\right)$$

~ solve for the large eddy averaged velocity and the modified pressure

~ subgrid stresses need modeling

10. LES

$$\bar{u}_i \left\{ \frac{\partial \bar{u}_i}{\partial t} + \frac{\partial(\bar{u}_i \bar{u}_j)}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(\frac{\bar{p}_{\text{mod}}}{\rho}\right) + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} + \bar{f}_i + \frac{\partial \tau_{ij}^{sub}}{\partial x_j} \right\}$$

$$\frac{\partial \frac{1}{2}\bar{u}_i^2}{\partial t} + \bar{u}_j \frac{\partial\left(\frac{1}{2}\bar{u}_i^2\right)}{\partial x_j}$$
$$= -\frac{\partial}{\partial x_i}\left(\frac{\bar{u}_i \bar{p}_{\text{mod}}}{\rho}\right) + \frac{\partial}{\partial x_j}\left(\nu \bar{u}_i \frac{\partial \bar{u}_i}{\partial x_j} + \bar{u}_i \tau_{ij}^{sub}\right) - \nu \frac{\partial \bar{u}_i}{\partial x_j}\frac{\partial \bar{u}_i}{\partial x_j} + \bar{u}_i \bar{f}_i - \tau_{ij}^{sub}\frac{\partial \bar{u}_i}{\partial x_j}$$

Energy exchange rate between resolved and subgrid eddies

$$\tau_{ij}^{sub}\frac{\partial \bar{u}_i}{\partial x_j}$$

10. LES

(a) Smagorinsky's model (1963, Mon. Weather Rev. 91, 99)
~ simplest, commonly used

• eddy viscosity: $\tau_{ij}^{sub} = -2\nu_t \bar{S}_{ij}$

$$\bar{S}_{ij} = \tfrac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

• mixing length assumption:

$$\nu_t \sim \Delta x \cdot \Delta u \sim \sigma \cdot \sigma\left|\bar{S}\right| \quad , \quad \left|\bar{S}\right| \equiv \left(2\bar{S}_{ij}\bar{S}_{ij}\right)^{1/2}$$

$$\nu_t = (C_S \sigma)^2 \left(2\bar{S}_{ij}\bar{S}_{ij}\right)^{1/2}$$

~ always positive (no backward cascade)

~ isotropic

~ incorrect near walls

10. LES

(a) Smagorinsky's model (1963, Mon. Weather Rev. 91, 99)

•Determination of the proportional constant:

Energy exchange rate between resolved and subgrid eddies:

$$<\varepsilon_g> = <\tau_{ij}^{sub}\frac{\partial \bar{u}_i}{\partial x_j}> = (C_S\sigma)^2 <|\bar{S}|^3>$$

Recall for homogeneous isotropic turbulence:

$$|\bar{S}| = <\omega_i\omega_i> \approx 2\int_0^{\pi/\sigma} k^2 E(k)dk$$

At large Reynolds number:

$$<|\bar{S}|^2> = <\omega_i\omega_i> \approx 2\int_0^{\pi/\sigma} k^2 E(k)dk \approx 2\int_0^{\pi/\sigma} k^2\left(C_K <\varepsilon>^{2/3} k^{-5/3}\right)dk$$

$$<\varepsilon> = \left(\frac{2}{3C_K}\right)^{3/2}\left(\frac{\sigma}{\pi}\right)^2 <|\bar{S}|^2>^{3/2}$$

10. LES

(a) Smagorinsky's model (1963, Mon. Weather Rev. 91, 99)

• Determination of the proportional constant:

At large Reynolds number: $<\varepsilon_g> \approx <\varepsilon>$

$$(C_S\sigma)^2 <|\bar{S}|^3> \approx \left(\frac{2}{3C_K}\right)^{3/2}\left(\frac{\sigma}{\pi}\right)^2 <|\bar{S}|^2>^{3/2}$$

$$C_S = \frac{1}{\pi}\left(\frac{2}{3C_K}\right)^{3/4}\frac{<|\bar{S}|^2>^{3/4}}{<|\bar{S}|^3>^{1/2}} \approx \frac{1}{\pi}\left(\frac{2}{3C_K}\right)^{3/4}$$

$$|\bar{S}| \equiv \left(2\bar{S}_{ij}\bar{S}_{ij}\right)^{1/2}$$

$$\nu_t = (C_S\sigma)^2\left(2\bar{S}_{ij}\bar{S}_{ij}\right)^{1/2}$$

$$\tau_{ij}^{sub} = -2\nu_t\bar{S}_{ij}$$

10. LES

(b) structure-function model (1992, JFM 239,157-194)

structure function: $F_2(\vec{x},\vec{r}) \equiv <|\vec{u}(\vec{x},t)-\vec{u}(\vec{x}+\vec{r},t)|^2>$

For homogeneous isotropic turbulence:

$$F_2(\vec{x},\vec{r}) = F_2(\vec{r}) = 2<u_i(\vec{x})u_i(\vec{x})> - 2<u_i(\vec{x})u_i(\vec{x}+\vec{r})> = 4\int_0^{\infty} E(k)\left(1-\frac{\sin kr}{kr}\right)dk$$

structure function of resolved eddies:

$$\bar{F}_2(r) \equiv <\left(\bar{u}_i(\vec{x})-\bar{u}_i(\vec{x}+\vec{r})\right)\left(\bar{u}_i(\vec{x})-\bar{u}_i(\vec{x}+\vec{r})\right)>$$

$$\bar{F}_2(r) \approx 4\int_0^{k_c} E(k)\left(1-\frac{\sin kr}{kr}\right)dk \quad , \quad k_c = \frac{\pi}{\Delta x}$$

$$\bar{F}_2(\Delta x) \approx 4\int_0^{k_c} E(k)\left(1-\frac{\sin k\Delta x}{k\Delta x}\right)dk$$

10. LES

For isotropic homogeneous turbulence:

$$<u_i(\vec{x})u_i(\vec{x}+\vec{r})> = \iiint \Phi_{ii}\left(\vec{k}\right)\exp\left(i\vec{k}\cdot\vec{r}\right)d\vec{k}$$

$$= \int_0^{k_{max}}\int_0^{\pi}\int_0^{2\pi}\Phi_{ii}(k)\exp(ikr\cos\theta)k^2\sin\theta d\phi d\theta dk$$

$$= 2\pi\int_0^{k_{max}} k^2\Phi_{ii}(k)\left(\int_0^{\pi}\exp(ikr\cos\theta)\sin\theta d\theta\right)dk$$

$$\int_0^{\pi}\exp(ikr\cos\theta)\sin\theta d\theta = -\left.\frac{\exp(ikr\cos\theta)}{ikr}\right|_0^{\pi} = \frac{\exp(ikr)}{ikr} - \frac{\exp(-ikr)}{ikr}$$

$$= \frac{2i\sin(kr)}{ikr} = \frac{2\sin(kr)}{kr}$$

10. LES

For isotropic homogeneous turbulence:

$$< u_i(\vec{x}) u_i(\vec{x}+\vec{r}) > = 4\pi \int_0^{k_{max}} k^2 \Phi_{ii}(k) \frac{\sin(kr)}{kr} dk$$

Recall $\quad \frac{1}{2}\overline{u_i u_i} = \frac{1}{2}\iiint \Phi_{ii}(\vec{k},t) d\vec{k} = \int_0^{k_{max}} E(k) dk$

$$E(k) = \int_0^{2\pi}\int_0^{\pi} \tfrac{1}{2}\Phi_{ii}(\vec{k},t) k^2 \sin\theta d\theta d\phi$$

$$= 2\pi k^2 \Phi_{ii}(\vec{k},t)$$

$$< u_i(\vec{x}) u_i(\vec{x}+\vec{r}) > = 2\int_0^{k_{max}} E(k) \frac{\sin(kr)}{kr} dk$$

10. LES

(b) structure-function model (1992, JFM 239,157-194)

At large Reynolds number:

$$\overline{F}_2(\Delta x) \approx 4\int_0^{k_c} E(k_c)\left(\frac{k}{k_c}\right)^{-5/3}\left(1-\frac{\sin k\Delta x}{k\Delta x}\right)dk = \frac{4\pi^{8/3}A}{\Delta x^2}\frac{E(k_c)}{k_c}$$

$$A \equiv \int_0^{\pi} \xi^{-5/3}\left(1-\frac{\sin\xi}{\xi}\right)d\xi \approx 0.476738$$

On the other hand, expect from mixing length assumption:

$$\nu_t \sim \Delta x \cdot \Delta u \sim k_c^{-1}\cdot(k_c E(k_c))^{1/2} = \left(\frac{E(k_c)}{k_c}\right)^{1/2}$$

$$\nu_t \equiv C_F\left(\frac{E(k_c)}{k_c}\right)^{1/2} = C_F\left\{\frac{\Delta x^2}{4\pi^{8/3}A}\overline{F}_2(\Delta x)\right\}^{1/2}$$

10. LES

(b) structure-function model (1992, JFM 239,157-194)

- Determination of proportional constant (C_F):

  At large Reynolds number: $< \varepsilon > \approx < \varepsilon_g > = 2\nu_t \int_0^{k_c} k^2 E(k) dk$

$$< \varepsilon > = 2\nu_t \int_0^{k_c} k^2 E(k) dk$$

$$= 2\nu_t \int_0^{k_c} k^2 \cdot C_K < \varepsilon >^{2/3} k^{-5/3} dk \qquad 1 = \frac{3}{2}\cdot C_F C_K^{3/2}$$

$$= 2\nu_t C_K < \varepsilon >^{2/3} \cdot \frac{3}{4} k_c^{4/3} \qquad C_F = \frac{3}{2} C_K^{-3/2}$$

$$1 = 2\nu_t C_K < \varepsilon >^{-1/3} \cdot \frac{3}{4} k_c^{4/3}$$

$$1 = \frac{3}{2}\cdot C_F \left(\frac{E(k_c)}{k_c}\right)^{1/2} \cdot \left(C_K < \varepsilon >^{2/3} \cdot k_c^{-5/3}\right)^{-1/2} \cdot C_K^{3/2} k_c^{1/2}$$

10. LES

(b) structure-function model (1992, JFM 239,157-194)

$$\nu_t \equiv C_F\left(\frac{E(k_c)}{k_c}\right)^{1/2} = C_F\left\{\frac{\Delta x^2}{4\pi^{8/3}A}\overline{F}_2(\Delta x)\right\}^{1/2}$$

$$C_F = \frac{3}{2} C_K^{-3/2}$$

$$\nu_t = \frac{2}{3} C_K^{-3/2}\left\{\frac{\Delta x^2}{4\pi^{8/3}A}\overline{F}_2(\Delta x)\right\}^{1/2} \approx 0.105 \cdot C_K^{-3/2}\cdot \Delta x \cdot \overline{F}_2^{1/2}(\Delta x)$$

- computation of structure function

$$\overline{F}_2(\vec{x},\Delta\vec{x}) \equiv < \left(\overline{u}_i(\vec{x}) - \overline{u}_i(\vec{x}+\Delta\vec{x})\right)\left(\overline{u}_i(\vec{x}) - \overline{u}_i(\vec{x}+\Delta\vec{x})\right) >$$

$$\approx \frac{1}{6}\left\{\begin{array}{l} +\left|\overline{u}_i(x_1)-\overline{u}_i(x_1+\Delta x_1)\right|^2 + \left|\overline{u}_i(x_1)-\overline{u}_i(x_1-\Delta x_1)\right|^2 \\ +\left|\overline{u}_i(x_2)-\overline{u}_i(x_2+\Delta x_2)\right|^2 + \left|\overline{u}_i(x_2)-\overline{u}_i(x_2-\Delta x_2)\right|^2 \\ +\left|\overline{u}_i(x_3)-\overline{u}_i(x_3+\Delta x_3)\right|^2 + \left|\overline{u}_i(x_3)-\overline{u}_i(x_3-\Delta x_3)\right|^2 \end{array}\right\}$$

10. LES

•Comparison between Smagorinsky's model and structure function model:

$$\Delta x_i = \Delta x$$

$$\left|\bar{u}_1(x_1) - \bar{u}_1(x_1+\Delta x_1)\right|^2 \approx \left|\frac{\partial \bar{u}_1}{\partial x_1}\Delta x\right|^2$$

$$\left|\bar{u}_2(x_1) - \bar{u}_2(x_1+\Delta x_1)\right|^2 \approx \left|\frac{\partial \bar{u}_2}{\partial x_1}\Delta x\right|^2$$

$$\bar{F}_2 \approx \frac{1}{6}\left\{\begin{array}{l} +\left|\bar{u}_i(x_1)-\bar{u}_i(x_1+\Delta x_1)\right|^2 + \left|\bar{u}_i(x_1)-\bar{u}_i(x_1-\Delta x_1)\right|^2 \\ +\left|\bar{u}_i(x_2)-\bar{u}_i(x_2+\Delta x_2)\right|^2 + \left|\bar{u}_i(x_2)-\bar{u}_i(x_2-\Delta x_2)\right|^2 \\ +\left|\bar{u}_i(x_3)-\bar{u}_i(x_3+\Delta x_3)\right|^2 + \left|\bar{u}_i(x_3)-\bar{u}_i(x_3-\Delta x_3)\right|^2 \end{array}\right\}$$

$$\bar{F}_2(\vec{x},\Delta\vec{x}) \approx \frac{1}{6}\left\{2\left(\frac{\partial \bar{u}_i}{\partial x_1}\Delta x\right)^2 + 2\left(\frac{\partial \bar{u}_i}{\partial x_2}\Delta x\right)^2 + 2\left(\frac{\partial \bar{u}_i}{\partial x_3}\Delta x\right)^2\right\}$$

10. LES

•Comparison between Smagorinsky's model and structure function model:

$$\bar{F}_2(\vec{x},\Delta x) = \frac{\Delta x^2}{3}\frac{\partial \bar{u}_i}{\partial x_j}\frac{\partial \bar{u}_i}{\partial x_j} = \frac{\Delta x^2}{3}\left(\bar{S}_{ij}\bar{S}_{ij} + \bar{\Omega}_{ij}\bar{\Omega}_{ij}\right) = \frac{\Delta x^2}{6}\left(2\bar{S}_{ij}\bar{S}_{ij} + \bar{\omega}_i\bar{\omega}_i\right)$$

$$\nu_t^F = 0.105 \cdot C_K^{-3/2} \cdot \Delta x \cdot \bar{F}_2^{1/2}(\Delta x) = 0.043 \cdot C_K^{-3/2} \cdot \Delta x^2 \cdot \sqrt{\left|\bar{S}\right|^2 + \bar{\omega}_i\bar{\omega}_i}$$

$$\nu_t^S = \left(\frac{1}{\pi}\left(\frac{2}{3C_K}\right)^{3/4}\sigma\right)^2 \left(2\bar{S}_{ij}\bar{S}_{ij}\right)^{1/2} = 0.055 \cdot C_K^{-3/2} \cdot \Delta x^2 \cdot \left|\bar{S}\right|$$

10. LES

(c) dynamic model (1992, JFM 238,352-336; 1991, Phys. Fluids A 3, 1760-1765)

idea: use two filters, one with width Δx and the other with $\alpha\Delta x$ (α>1)

$$\bar{u}_i(\vec{x},t) \equiv \int u_i(\vec{y},t)G(|\vec{y}-\vec{x}|;\Delta x)d\vec{y}$$

$$\tilde{u}_i(\vec{x},t) \equiv \int u_i(\vec{y},t)G(|\vec{y}-\vec{x}|;\alpha\Delta x)d\vec{y}$$

$$\tilde{\bar{u}}_i(\vec{x},t) \equiv \int \bar{u}_i(\vec{y},t)G(|\vec{y}-\vec{x}|;\alpha\Delta x)d\vec{y}$$

$$\frac{\partial \bar{u}_j}{\partial x_j} = 0$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial(\bar{u}_i\bar{u}_j)}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(\frac{\bar{p}}{\rho}\right) + \nu\frac{\partial \bar{u}_i}{\partial x_j\partial x_j} + \bar{f}_i + \frac{\partial T_{ij}}{\partial x_j}$$

$$T_{ij} \equiv \bar{u}_i\bar{u}_j - \overline{u_i u_j}$$

10. LES

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial(\bar{u}_i\bar{u}_j)}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(\frac{\bar{p}}{\rho}\right) + \nu\frac{\partial \bar{u}_i}{\partial x_j\partial x_j} + \bar{f}_i + \frac{\partial T_{ij}}{\partial x_j}$$

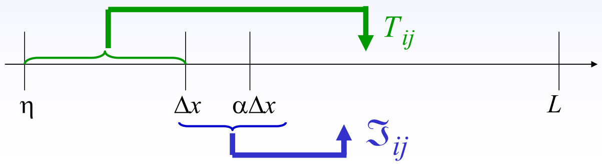
$$\frac{\partial \tilde{\bar{u}}_i}{\partial t} + \frac{\partial(\tilde{\bar{u}}_i\tilde{\bar{u}}_j)}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(\frac{\tilde{\bar{p}}}{\rho}\right) + \nu\frac{\partial \tilde{\bar{u}}_i}{\partial x_j\partial x_j} + \tilde{\bar{f}}_i + \frac{\partial \Im_{ij}}{\partial x_j}$$

$$\Im_{ij} \equiv \tilde{T}_{ij} + \tilde{\bar{u}}_i\tilde{\bar{u}}_j - \widetilde{\bar{u}_i\bar{u}_j}$$

$$= \tilde{T}_{ij} - \tilde{L}_{ij}$$

$$\tilde{L}_{ij} \equiv \widetilde{\bar{u}_i\bar{u}_j} - \tilde{\bar{u}}_i\tilde{\bar{u}}_j$$

$$\tilde{T}_{ij} \equiv \widetilde{\bar{u}_i\bar{u}_j} - \widetilde{\overline{u_i u_j}}$$



10. LES

Germano identity: $\tilde{L}_{ij} = \tilde{T}_{ij} - \Im_{ij}$

If Smagorinsky's model is applied, then

$$T_{ij} - \tfrac{1}{3} T_{kk}\delta_{ij} = 2C\Delta x^2 |\bar{S}|\bar{S}_{ij}$$

= effect of eddies of size $< \Delta x$ on those of size $> \Delta x$

$$\Im_{ij} - \tfrac{1}{3}\Im_{kk}\delta_{ij} = 2C(\alpha\Delta x)^2 |\tilde{\bar{S}}|\tilde{\bar{S}}_{ij}$$

= effect of eddies of size $< \alpha\Delta x$ on those of size $> \alpha\Delta x$

$$\tilde{\bar{S}}_{ij} = \tfrac{1}{2}\left(\frac{\partial \tilde{\bar{u}}_i}{\partial x_j} + \frac{\partial \tilde{\bar{u}}_j}{\partial x_i}\right)$$

$$\tilde{L}_{ij} - \tfrac{1}{3}\tilde{L}_{kk}\delta_{ij} = 2\left(\widetilde{C\Delta x^2 |\bar{S}|\bar{S}_{ij}} - C\alpha^2\Delta x^2 |\tilde{\bar{S}}|\tilde{\bar{S}}_{ij}\right)$$

10. LES

Assume C = constant:

$$\tilde{L}_{ij} - \tfrac{1}{3}\tilde{L}_{kk}\delta_{ij} = 2C\Delta x^2\left(\widetilde{|\bar{S}|\bar{S}_{ij}} - \alpha^2 |\tilde{\bar{S}}|\tilde{\bar{S}}_{ij}\right) \equiv 2CM_{ij}$$

- One possibility:

$$\bar{S}_{ij}\left(\tilde{L}_{ij} - \tfrac{1}{3}\tilde{L}_{kk}\delta_{ij}\right) = 2CM_{ij}\bar{S}_{ij}$$

$$C = \frac{1}{2}\frac{\tilde{L}_{ij}\bar{S}_{ij}}{M_{mn}\bar{S}_{mn}}$$

- Another possibility: choose C to be the one that minimizes $E(C)$

$$E(C) \equiv \left(\tilde{L}_{ij} - \tfrac{1}{3}\tilde{L}_{kk}\delta_{ij} - 2CM_{ij}\right)\left(\tilde{L}_{ij} - \tfrac{1}{3}\tilde{L}_{kk}\delta_{ij} - 2CM_{ij}\right)$$

$$\frac{\partial E(C)}{\partial C} = 0 = -2M_{ij}\left(\tilde{L}_{ij} - \tfrac{1}{3}\tilde{L}_{kk}\delta_{ij} - 2CM_{ij}\right)$$

10. LES

$$0 = -2M_{ij}\left(\tilde{L}_{ij} - \tfrac{1}{3}\tilde{L}_{kk}\delta_{ij} - 2CM_{ij}\right)$$

$$0 = M_{ij}\tilde{L}_{ij} - \tfrac{1}{3}M_{ij}\tilde{L}_{kk}\delta_{ij} - 2CM_{ij}M_{ij}$$

$$M_{ij}\tilde{L}_{ij} - \cancel{\tfrac{1}{3}M_{jj}\tilde{L}_{kk}} = 2CM_{ij}M_{ij}$$

$$M_{ij} \equiv \Delta x^2\left(|\bar{S}|\bar{S}_{ij} - \alpha^2 |\tilde{\bar{S}}|\tilde{\bar{S}}_{ij}\right)$$

$$C = \frac{1}{2}\frac{\tilde{L}_{ij}M_{ij}}{M_{mn}M_{mn}}$$

10. LES

$$\tilde{\bar{S}}_{ij} = \tfrac{1}{2}\left(\frac{\partial \tilde{\bar{u}}_i}{\partial x_j} + \frac{\partial \tilde{\bar{u}}_j}{\partial x_i}\right)$$

$$M_{ij} \equiv \Delta x^2\left(|\bar{S}|\bar{S}_{ij} - \alpha^2 |\tilde{\bar{S}}|\tilde{\bar{S}}_{ij}\right)$$

$$\tilde{L}_{ij} \equiv \bar{u}_i\bar{u}_j - \tilde{\bar{u}}_i\tilde{\bar{u}}_j$$

$$C = \frac{1}{2}\frac{\tilde{L}_{ij}M_{ij}}{M_{mn}M_{mn}}$$

$$T_{ij} - \tfrac{1}{3}T_{kk}\delta_{ij} = 2C\Delta x^2 |\bar{S}|\bar{S}_{ij} \qquad \textbf{(subgrid stresses)}$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial(\bar{u}_i\bar{u}_j)}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(\frac{\bar{p}}{\rho}\right) + \nu\frac{\partial \bar{u}_i}{\partial x_j\partial x_j} + \bar{f}_i + \frac{\partial T_{ij}}{\partial x_j}$$

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

10. LES

$$C = \frac{1}{2}\frac{\tilde{L}_{ij} M_{ij}}{M_{mn} M_{mn}} = C(\vec{x}, t)$$

dynamic model:

- C = 0 in laminar flows and at walls.
- A negative C is possible (backward cascade).
- mathematically inconsistent (dynamic localization methods)
- Too strong negativeness causes instability.
- The real C is found to be intermittent.

Other subgrid models:

- variations of dynamic model

(1992) Phys. Fluids A 4, 633-635

(1994) Center for Turbulence, Proc. Of the Summer Program 271-299

(1995) JFM 286, 229-255

- k-equation dynamic model $T_{ij} - \frac{1}{3} T_{kk} \delta_{ij} = 2C\Delta x \cdot k^{1/2} \cdot \overline{S}_{ij}$

10. LES

(d) spectral models (models in the wave space)

$$\frac{\partial E(k)}{\partial t} = T(k) - 2\nu k^2 E(k)$$

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) E(k) = T(k) = \int_{|\vec{k}|=k} \left(\iint_{\vec{p}+\vec{q}+\vec{k}=0} S(\vec{k}; \vec{p}, \vec{q}) d\vec{p} d\vec{q} \right) d\vec{k}$$

$$\equiv \iint_{\vec{p}+\vec{q}+\vec{k}=0} T(k; p, q) d\vec{p} d\vec{q}$$

$$= \iint_{\substack{\vec{p}+\vec{q}+\vec{k}=0 \\ p \text{ and } q<k_c}} T(k; p, q) d\vec{p} d\vec{q} + \iint_{\substack{\vec{p}+\vec{q}+\vec{k}=0 \\ p \text{ or } q>k_c}} T(k; p, q) d\vec{p} d\vec{q}$$

cascade due to subgrid stresses, to be modeled

$$\begin{cases} k < k_c : & \text{resolved eddies} \\ k > k_c : & \text{subgrid eddies} \end{cases}$$

10. LES

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) E(k) = \iint_{\substack{\vec{p}+\vec{q}+\vec{k}=0 \\ p \text{ and } q<k_c}} T(k; p, q) d\vec{p} d\vec{q} + \iint_{\substack{\vec{p}+\vec{q}+\vec{k}=0 \\ p \text{ or } q>k_c}} T(k; p, q) d\vec{p} d\vec{q}$$

$$= \iint_{\substack{\vec{p}+\vec{q}+\vec{k}=0 \\ p \text{ and } q<k_c}} T(k; p, q) d\vec{p} d\vec{q} + 2\nu_t k^2 E(k)$$

$$\left(\frac{\partial}{\partial t} + 2(\nu + \nu_t) k^2\right) E(k) = \iint_{\substack{\vec{p}+\vec{q}+\vec{k}=0 \\ p \text{ and } q<k_c}} T(k; p, q) d\vec{p} d\vec{q}$$

$$2\nu_t k^2 E(k) = \iint_{\substack{\vec{p}+\vec{q}+\vec{k}=0 \\ p \text{ or } q>k_c}} T(k; p, q) d\vec{p} d\vec{q}$$

10. LES

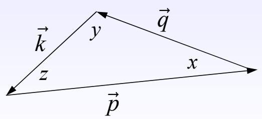
$$2\nu_t k^2 E(k) = \iint_{\substack{\vec{p}+\vec{q}+\vec{k}=0 \\ p \text{ or } q>k_c}} T(k; p, q) d\vec{p} d\vec{q}$$

Isotropic homogeneous turbulence theory:

EDQNM (Eddy-damped quasi-normal Markovian by Orszag)

DIA (direct interaction approximation by Kraichnan)

$$T(k, p, q) = \theta_{kpq} \cdot \frac{xy + z^3}{q} \cdot E(q) \cdot \left(k^2 E(p) - p^2 E(k)\right)$$



x,y,z : cosine value of the corresponding angles

$$E(k) = C_K \langle \varepsilon \rangle^{2/3} k^{-5/3}$$

$$\theta_{kpq} = \left(\eta_k + \eta_p + \eta_q\right)^{-1}$$

$$\eta_k = \mu C_K \langle \varepsilon \rangle^{1/3} k^{2/3}$$

10. LES

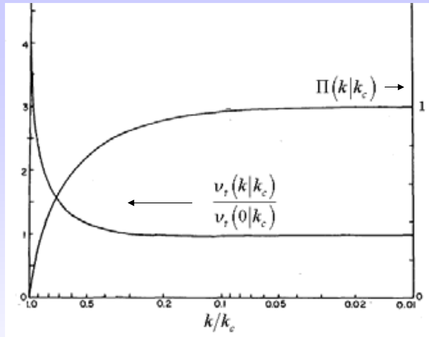
$$\left(\frac{\partial}{\partial t}+\nu k^2\right)\hat{u}_i = -i\Delta_{ij}k_m \iint_{\vec{p}+\vec{q}-\vec{k}=0} \hat{u}_j(\vec{p})\hat{u}_m(\vec{q})d\vec{p}d\vec{q}$$

$$\left(\frac{\partial}{\partial t}+(\nu+\nu_t)k^2\right)\hat{u}_i = -i\Delta_{ij}k_m \iint_{\substack{\vec{p}+\vec{q}=\vec{k}\\ p,q<k_c}} \hat{u}_j(\vec{p})\hat{u}_m(\vec{q})d\vec{p}d\vec{q}$$

for $k<k_c$

Kraichnan,1976, Eddy viscosity in two and three dimensions,
J. Atmos. Sci. 33 1521–36.

10. LES



$$\Pi(k|k_c) \equiv -\frac{\int_k^{k_c} T(k'|k_c)dk'}{\int_0^{k_c} T(k'|k_c)dk'}$$ = the fraction of the total energy transfer across k_c which comes from wavenumbers between k and k_c

10. LES

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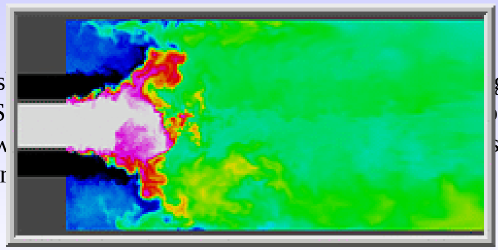

10. LES

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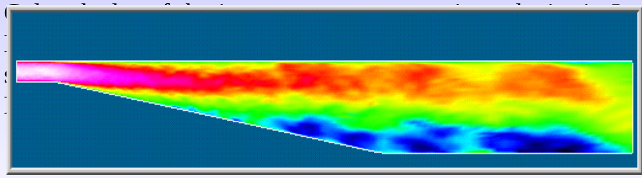

10. LES

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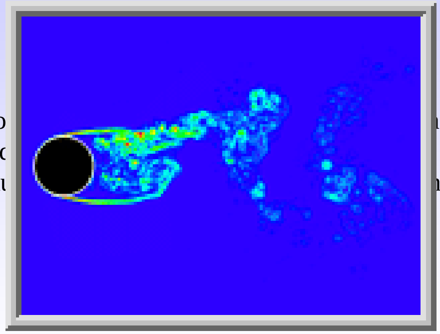


10. DNS, LES, and RANS

RANS (Reynolds Averaged Navier-Stokes Equations)

$$\frac{\partial \bar{u}_j}{\partial x_j} = 0$$

$$\frac{D\bar{u}_i}{Dt} \equiv \frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = g_i - \frac{\partial}{\partial x_i}\left(\frac{\bar{p}}{\rho} + \frac{2}{3}K\right) + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\bar{\tau}_{ij} + \tau_{ij}^t\right)$$

(Here overbar represents an ensemble average.)

$$\bar{\tau}_{ij} = \mu\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$ mean viscous stress tensor

$$\tau_{ij}^t = -\rho \overline{u_i' u_j'} + \frac{2}{3}\rho K \delta_{ij}$$ = Reynolds (turbulent) stress tensor

- Zero equation
- One equation Model : model K-equation
- Two equation Model : K – equation + ε – equation
- Reynolds stress models: model τ_{ij}^t -equations

10. RANS

§ Zero equation

- **Eddy Viscosity Concept**: $\frac{\tau_{ij}^t}{\rho} = 2\nu_t \bar{S}_{ij}$ (divergence - free)

  ν_t : eddy viscosity (isotropic herein)

- **Mixing length models**: $\nu_t \approx l^2 |\bar{S}|$

¶ Prandtl and Karman:

Sublayer: $l \approx y^2$

Overlap layer: $l \approx \kappa y$

Outerlayer: $l \approx$ constant

¶ van Driest Model

$$l \approx \kappa y \underbrace{\left[1 - \exp\left(-\frac{y^+}{A}\right)\right]}_{\text{damping factor}} ;\ A \approx 26 \text{ for flat-plate flow}$$

A varies with flow conditions
(pressure gradient, wall roughness, blowing/suction etc)

10. RANS

§ One-Equation Model

- **Eddy Viscosity Concept**: $\frac{\tau_{ij}^t}{\rho} = 2\nu_t \bar{S}_{ij}$ (divergence - free)

- dimensional analysis:

$$\nu_t = f(K, \bar{\varepsilon}) \quad = C_\mu K^2/\bar{\varepsilon}$$

ν_t: m^2/s; K: m^2/s^2; $\bar{\varepsilon}$: m^2/s^3

- turbulent kinetic energy dissipation rate:

$$\bar{\varepsilon} = \frac{\text{drag} \times \text{velocity}}{\text{mass}} \sim \frac{\left|\rho \times \text{velocity}^2 \times \text{area}\right| \times \text{velocity}}{\rho L^3} \sim \frac{K^{3/2}}{L}$$

$K^{1/2} \propto$	eddy velocity
$L \propto$	effective eddy size

$$\bar{\varepsilon} = C_\varepsilon \frac{K^{3/2}}{L}$$

10. RANS

• turbulent kinetic energy per unit mass K:

$$\rho\frac{DK}{Dt}=\rho\frac{\partial K}{\partial t}+\rho\bar{u}_j\frac{\partial K}{\partial x_j}$$

$$=\frac{\partial}{\partial x_j}\left\{-\overline{p'u'_j}-\tfrac{1}{2}\rho\overline{u'_iu'_iu'_j}+\mu\frac{\partial K}{\partial x_j}\right\}-\rho\overline{u'_iu'_j}\frac{\partial\bar{u}_i}{\partial x_j}-2\mu\underbrace{\overline{S'_{ij}S'_{ij}}}_{=\bar{\varepsilon}}$$

- turbulent diffusion term:

$$-\frac{1}{\rho}\overline{p'u'_j}-\tfrac{1}{2}\overline{u'_iu'_iu'_j}\;\cong\;C_K\left[\frac{\ell^2}{t}\right]\frac{\partial K}{\partial x_j}\;:\equiv C_K\frac{K^2}{\bar{\varepsilon}}\frac{\partial K}{\partial x_j}$$

- turbulent production term:

$$-\rho\overline{u'_iu'_j}\frac{\partial\bar{u}_i}{\partial x_j}=\left(\tau^t_{ij}-\tfrac{2}{3}\rho K\delta_{ij}\right)\frac{\partial\bar{u}_i}{\partial x_j}=\tau^t_{ij}\frac{\partial\bar{u}_i}{\partial x_j}$$

$$\frac{\tau^t_{ij}}{\rho}=2\nu_t\bar{S}_{ij}=2C_\mu\frac{K^2}{\bar{\varepsilon}}\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)$$

10. RANS

§ **One-Equation Model**

$$\frac{\partial K}{\partial t}+\bar{u}_j\frac{\partial K}{\partial x_j}=\frac{\partial}{\partial x_j}\left\{C_K\frac{K^2}{\bar{\varepsilon}}\frac{\partial K}{\partial x_j}+\nu\frac{\partial K}{\partial x_j}\right\}+C_\mu\frac{K^2}{\bar{\varepsilon}}\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)\frac{\partial\bar{u}_i}{\partial x_j}-\bar{\varepsilon}$$

$$\frac{\partial\bar{u}_j}{\partial x_j}=0$$

$$\frac{\partial\bar{u}_i}{\partial t}+\bar{u}_j\frac{\partial\bar{u}_i}{\partial x_j}=g_i-\frac{\partial}{\partial x_i}\left(\frac{\bar{p}}{\rho}+\tfrac{2}{3}K\right)+\frac{\partial}{\partial x_j}\left(\nu\frac{\partial\bar{u}_i}{\partial x_j}+\overbrace{2C_\mu\frac{K^2}{\bar{\varepsilon}}\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)}^{\tau^t_{ij}/\rho}\right)$$

$$\bar{\varepsilon}=C_\varepsilon\frac{K^{3/2}}{L}$$

6 equations for 6 unknowns $\left(\bar{u}_i,\ \frac{\bar{p}}{\rho}+\tfrac{2}{3}K,\ K,\ \bar{\varepsilon}\right)$

with 3 empirical constants $(C_K,\ C_\mu,\ C_\varepsilon/L)$

10. RANS

§ **Two-Equation Model**

• turbulent kinetic energy dissipation rate (per unit mass): $\bar{\varepsilon}\equiv\nu\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_i}{\partial x_j}}$

exact equation:

$$\frac{D\bar{\varepsilon}}{Dt}=\frac{\partial}{\partial x_j}\left\{\underbrace{-\nu\overline{u'_j\frac{\partial u'_i}{\partial x_m}\frac{\partial u'_i}{\partial x_m}}-\frac{2\nu}{\rho}\overline{\frac{\partial u'_j}{\partial x_m}\frac{\partial p'}{\partial x_m}}}_{\text{turbulent diffusion}}+\underbrace{\nu\frac{\partial\bar{\varepsilon}}{\partial x_j}}_{\text{molecular diffusion}}\right\}$$

$$\underbrace{-2\nu\overline{u'_j\frac{\partial u'_i}{\partial x_m}}\frac{\partial^2\bar{u}_i}{\partial x_j\partial x_m}-2\nu\frac{\partial\bar{u}_i}{\partial x_m}\left\{\overline{\frac{\partial u'_j}{\partial x_i}\frac{\partial u'_j}{\partial x_m}}+\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_m}{\partial x_j}}\right\}}_{\text{production}}$$

$$\underbrace{-2\nu\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_i}{\partial x_m}\frac{\partial u'_j}{\partial x_m}}-2\nu\overline{\frac{\partial^2u'_i}{\partial x_j\partial x_j}\frac{\partial^2u'_i}{\partial x_m\partial x_m}}}_{\text{destruction}}$$

10. RANS

- turbulent diffusion terms:

$$-\nu\overline{u'_j\frac{\partial u'_i}{\partial x_m}\frac{\partial u'_i}{\partial x_m}}-\frac{2\nu}{\rho}\overline{\frac{\partial u'_j}{\partial x_m}\frac{\partial p'}{\partial x_m}}\cong\left[\frac{m^2}{s}\right]\frac{\partial\bar{\varepsilon}}{\partial x_j}\;:\equiv C_\varepsilon\frac{K^2}{\bar{\varepsilon}}\frac{\partial\bar{\varepsilon}}{\partial x_j}$$

- production terms:

$$-2\nu\overline{u'_j\frac{\partial u'_i}{\partial x_m}}\frac{\partial^2\bar{u}_i}{\partial x_j\partial x_m}-2\nu\frac{\partial\bar{u}_i}{\partial x_m}\left\{\overline{\frac{\partial u'_j}{\partial x_i}\frac{\partial u'_j}{\partial x_m}}+\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_m}{\partial x_j}}\right\}$$

$$\cong\left[\frac{m^3}{kg\cdot s}\right]\cdot\tau^t_{ij}\frac{\partial\bar{u}_i}{\partial x_j}\quad:\equiv -C_{\varepsilon1}\frac{\bar{\varepsilon}}{K}\overline{u'_iu'_j}\frac{\partial\bar{u}_i}{\partial x_j}$$

$$:\equiv 2C_\mu C_{\varepsilon1}K\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)\frac{\partial\bar{u}_i}{\partial x_j}$$

- destruction terms:

$$-2\nu\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_i}{\partial x_m}\frac{\partial u'_j}{\partial x_m}}-2\nu\overline{\frac{\partial^2u'_i}{\partial x_j\partial x_j}\frac{\partial^2u'_i}{\partial x_m\partial x_m}}\cong\left[\frac{1}{\text{sec}}\right]\bar{\varepsilon}\quad:\equiv -C_{\varepsilon2}\frac{\bar{\varepsilon}}{K}\cdot\bar{\varepsilon}$$

10. RANS

§ **Two-Equation Model**

$$\frac{\partial K}{\partial t}+\bar{u}_j\frac{\partial K}{\partial x_j}=\frac{\partial}{\partial x_j}\left\{C_K\frac{K^2}{\varepsilon}\frac{\partial K}{\partial x_j}+\nu\frac{\partial K}{\partial x_j}\right\}+C_\mu\frac{K^2}{\varepsilon}\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j}-\varepsilon$$

$$\frac{\partial \bar{u}_j}{\partial x_j}=0$$

$$\frac{\partial \bar{u}_i}{\partial t}+\bar{u}_j\frac{\partial \bar{u}_i}{\partial x_j}=g_i-\frac{\partial}{\partial x_i}\left(\frac{\bar{p}}{\rho}+\frac{2}{3}K\right)+\frac{\partial}{\partial x_j}\left(\nu\frac{\partial \bar{u}_i}{\partial x_j}+2C_\mu\frac{K^2}{\varepsilon}\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)\right)$$

$$\frac{D\varepsilon}{Dt}=\frac{\partial}{\partial x_l}\left\{C_\varepsilon\frac{K^2}{\varepsilon}\frac{\partial \varepsilon}{\partial x_l}+\nu\frac{\partial \varepsilon}{\partial x_l}\right\}+2C_\mu C_{\varepsilon 1}K\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j}-C_{\varepsilon 2}\frac{\varepsilon^2}{K}$$

6 equations for 6 unknowns $\left(\bar{u}_i,\ \frac{\bar{p}}{\rho}+\frac{2}{3}K,\ K,\ \varepsilon\right)$

with 5 empirical constants $\left(C_K,\ C_\mu,\ C_\varepsilon, C_{\varepsilon 1}, C_{\varepsilon 2}\right)$

10. RANS

§ **Reynolds-stress Model**

RANS (Reynolds Averaged Navier-Stokes Equations)

$$\frac{\partial \bar{u}_j}{\partial x_j}=0$$

$$\frac{D\bar{u}_i}{Dt}\equiv\frac{\partial \bar{u}_i}{\partial t}+\bar{u}_j\frac{\partial \bar{u}_i}{\partial x_j}=g_i-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i}+\frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\bar{\tau}_{ij}+\tau^t_{ij}\right)$$

$$\bar{\tau}_{ij}=\mu\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)$$ mean viscous stress tensor

$\tau^t_{ij}=-\rho\,\overline{u'_i u'_j}$ = Reynolds (turbulent) stress tensor

Model τ^t_{ij} equations directly.

10. RANS

Reynolds stress tensor equations

$$\frac{D\overline{u'_i u'_j}}{Dt}=\frac{\partial \overline{u'_i u'_j}}{\partial t}+\bar{u}_m\frac{\partial \overline{u'_i u'_j}}{\partial x_m}$$ ~ mean motion Lagrangian

$$=-\frac{\partial}{\partial x_m}\left\{\overline{\left(u'_j\delta_{im}+u'_i\delta_{jm}\right)\frac{p'}{\rho}}\right\}+\overline{\frac{p'}{\rho}\left(\frac{\partial u'_j}{\partial x_i}+\frac{\partial u'_i}{\partial x_j}\right)}$$ (nonlocal, linear, and nonlinear)

$$+\nu\left(\frac{\partial^2\overline{u'_i u'_j}}{\partial x_m\partial x_m}-2\overline{\frac{\partial u'_i}{\partial x_m}\frac{\partial u'_j}{\partial x_m}}\right)$$ ~ viscous diffusion/dissipation effect

→ nonlocal

$$-\left(\overline{u'_i u'_m}\frac{\partial \bar{u}_j}{\partial x_m}+\overline{u'_j u'_m}\frac{\partial \bar{u}_i}{\partial x_m}\right)$$ ~ production and reorientation by the mean motion

$$-\frac{\partial \overline{u'_i u'_j u'_m}}{\partial x_m}$$ ~ turbulent advection

10. RANS

- turbulent diffusion terms:

$$-\overline{\left(u'_j\delta_{im}+u'_i\delta_{jm}\right)\frac{p'}{\rho}}-\overline{u'_i u'_j u'_m}\ \cong\left[\frac{m^2}{\sec}\right]\frac{\partial \overline{u'_i u'_j}}{\partial x_m}\ :=C_K\frac{K^2}{\varepsilon}\cdot\frac{\partial \overline{u'_i u'_j}}{\partial x_m}$$

- pressure-strain terms:

$$\overline{\frac{p'}{\rho}\left(\frac{\partial u'_j}{\partial x_i}+\frac{\partial u'_i}{\partial x_j}\right)}$$ traceless, expected to be able to be expressed in terms of $\frac{\partial \bar{u}_i}{\partial x_j}$ and $-\overline{u'_i u'_j}$

$$\cong\left[-\overline{u'_i u'_j}\right]\left[\frac{\partial \bar{u}_i}{\partial x_j}\right]\ :=C_2\left\{\overline{u'_i u'_m}\frac{\partial \bar{u}_j}{\partial x_m}+\overline{u'_j u'_m}\frac{\partial \bar{u}_i}{\partial x_m}-\frac{2}{3}\delta_{ij}\overline{u'_n u'_m}\frac{\partial \bar{u}_n}{\partial x_m}\right\}$$

- dissipation terms:

non-isotropic part

$$-2\nu\overline{\frac{\partial u'_i}{\partial x_m}\frac{\partial u'_j}{\partial x_m}}\ :=-\frac{2}{3}\delta_{ij}\varepsilon-C_1\frac{\varepsilon}{K}\left(\overline{u'_i u'_j}-\frac{2}{3}\delta_{ij}K\right)$$

isotropic part

10. RANS

modeled Reynolds stress tensor equations

$$\frac{D\overline{u'_i u'_j}}{Dt} = \frac{\partial}{\partial x_m}\left\{\left(C_K \frac{K^2}{\bar{\varepsilon}} + \nu\right)\frac{\partial \overline{u'_i u'_j}}{\partial x_m}\right\} - \left\{\overline{u'_i u'_m}\frac{\partial \bar{u}_j}{\partial x_m} + \overline{u'_j u'_m}\frac{\partial \bar{u}_i}{\partial x_m}\right\}$$
$$+ C_2\left\{\overline{u'_i u'_m}\frac{\partial \bar{u}_j}{\partial x_m} + \overline{u'_j u'_m}\frac{\partial \bar{u}_i}{\partial x_m} - \frac{2}{3}\delta_{ij}\overline{u'_n u'_m}\frac{\partial \bar{u}_n}{\partial x_m}\right\}$$
$$- \frac{2}{3}\delta_{ij}\bar{\varepsilon} - C_1\frac{\bar{\varepsilon}}{K}\left(\overline{u'_i u'_j} - \frac{2}{3}\delta_{ij}K\right)$$

6 equations for 6 new unknowns $\left(\overline{u_1'^2},\ \overline{u_2'^2},\ \overline{u_3'^2},\ \overline{u'_1 u'_2},\ \overline{u'_2 u'_3},\ \overline{u'_3 u'_1}\right)$

i=j :

$$\frac{DK}{Dt} = \frac{\partial}{\partial x_m}\left\{\left(C_K \frac{K^2}{\bar{\varepsilon}} + \nu\right)\frac{\partial K}{\partial x_m}\right\} - \frac{1}{2}\left\{\overline{u'_i u'_m}\frac{\partial \bar{u}_i}{\partial x_m} + \overline{u'_i u'_m}\frac{\partial \bar{u}_i}{\partial x_m}\right\} - \bar{\varepsilon}$$

10. RANS

§ Reynolds-stress Model

$$\frac{D\bar{\varepsilon}}{Dt} = \frac{\partial}{\partial x_l}\left\{C_\varepsilon \frac{K^2}{\bar{\varepsilon}}\frac{\partial \bar{\varepsilon}}{\partial x_l} + \nu\frac{\partial \bar{\varepsilon}}{\partial x_l}\right\} - C_{\varepsilon 1}\frac{\bar{\varepsilon}}{K}\overline{u'_i u'_j}\frac{\partial \bar{u}_i}{\partial x_j} - C_{\varepsilon 2}\frac{\bar{\varepsilon}^2}{K}$$

$$\frac{\partial \bar{u}_j}{\partial x_j} = 0$$

11 equations for 11 unknowns
$\left(\bar{u}_1, \bar{u}_2, \bar{u}_3, \frac{\bar{p}}{\rho}, \bar{\varepsilon}, \overline{u_1'^2},\ \overline{u_2'^2},\ \overline{u_3'^2},\ \overline{u'_1 u'_2},\ \overline{u'_2 u'_3},\ \overline{u'_3 u'_1}\right)$

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j\frac{\partial \bar{u}_i}{\partial x_j} = g_i - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial \bar{u}_i}{\partial x_j} - \overline{u'_i u'_j}\right)$$

$$\frac{D\overline{u'_i u'_j}}{Dt} = \frac{\partial}{\partial x_m}\left\{\left(C_K \frac{K^2}{\bar{\varepsilon}} + \nu\right)\frac{\partial \overline{u'_i u'_j}}{\partial x_m}\right\} - \left\{\overline{u'_i u'_m}\frac{\partial \bar{u}_j}{\partial x_m} + \overline{u'_j u'_m}\frac{\partial \bar{u}_i}{\partial x_m}\right\}$$
$$+ C_2\left\{\overline{u'_i u'_m}\frac{\partial \bar{u}_j}{\partial x_m} + \overline{u'_j u'_m}\frac{\partial \bar{u}_i}{\partial x_m} - \frac{2}{3}\delta_{ij}\overline{u'_n u'_m}\frac{\partial \bar{u}_n}{\partial x_m}\right\}$$
$$- \frac{2}{3}\delta_{ij}\bar{\varepsilon} - C_1\frac{\bar{\varepsilon}}{K}\left(\overline{u'_i u'_j} - \frac{2}{3}\delta_{ij}K\right)$$

10. RANS

§ Algebraic Stress Model

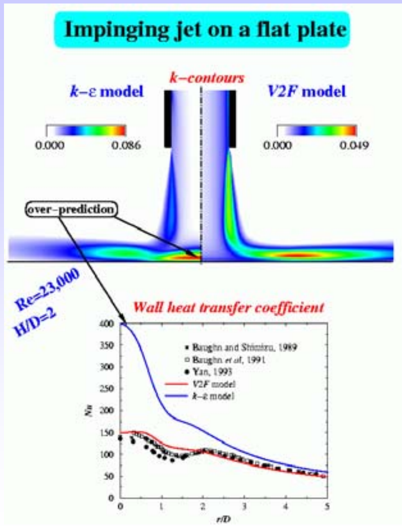
Assume negligible turbulent convection and diffusion

$$\frac{D\overline{u'_i u'_j}}{Dt} = \frac{\partial}{\partial x_m}\left\{\left(C_K \frac{K^2}{\bar{\varepsilon}} + \nu\right)\frac{\partial \overline{u'_i u'_j}}{\partial x_m}\right\} - \left\{\overline{u'_i u'_m}\frac{\partial \bar{u}_j}{\partial x_m} + \overline{u'_j u'_m}\frac{\partial \bar{u}_i}{\partial x_m}\right\}$$
$$+ C_2\left\{\overline{u'_i u'_m}\frac{\partial \bar{u}_j}{\partial x_m} + \overline{u'_j u'_m}\frac{\partial \bar{u}_i}{\partial x_m} - \frac{2}{3}\delta_{ij}\overline{u'_n u'_m}\frac{\partial \bar{u}_n}{\partial x_m}\right\}$$
$$- \frac{2}{3}\delta_{ij}\bar{\varepsilon} - C_1\frac{\bar{\varepsilon}}{K}\left(\overline{u'_i u'_j} - \frac{2}{3}\delta_{ij}K\right)$$

$$0 = -\left\{\overline{u'_i u'_m}\frac{\partial \bar{u}_j}{\partial x_m} + \overline{u'_j u'_m}\frac{\partial \bar{u}_i}{\partial x_m}\right\} - \frac{2}{3}\delta_{ij}\bar{\varepsilon} - C_1\frac{\bar{\varepsilon}}{K}\left(\overline{u'_i u'_j} - \frac{2}{3}\delta_{ij}K\right)$$
$$+ C_2\left\{\overline{u'_i u'_m}\frac{\partial \bar{u}_j}{\partial x_m} + \overline{u'_j u'_m}\frac{\partial \bar{u}_i}{\partial x_m} - \frac{2}{3}\delta_{ij}\overline{u'_n u'_m}\frac{\partial \bar{u}_n}{\partial x_m}\right\}$$

~ algebraic equations for the Reynolds stresses ~

10. RANS



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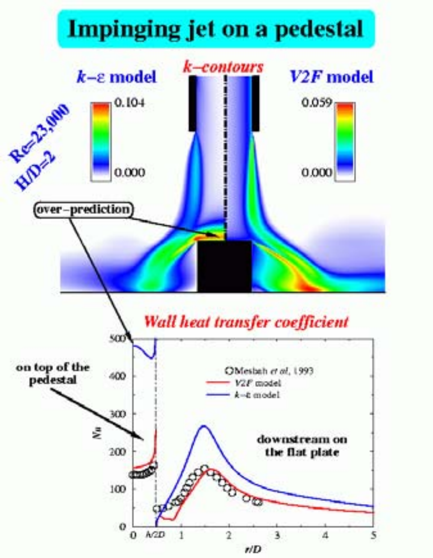
10. RANS



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10. DNS, LES, and RANS

Comparisons among DNS, LES and RANS:

DNS

• simulate motion of all scales (limited only by numerical errors)

• able to capture all details of the turbulent flow fields

• tremendously large amount of computations and huge data size

LES

• simulate motion of large eddies only (less computations)

• need modeling effects of subgrid eddies on large eddies

• able to see fluctuations over length scale >= Δx

RANS

• compute only ensemble averaged quantities (least computations)

• need strong assumptions to model effects of fluctuations on the mean motion